Orthogonal polynomials of several discrete variables and the 3nj-Wigner symbols: applications to spin networks

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Abstract. The use of orthogonal polynomials for integral models on the lattice is applied to the 3nj-symbols that appear in the coupling of several angular momenta. These symbols are connected to the Ponzano-Regge method to solve the Einstein equations on a discrete Riemannian manifold.

1. Classical orthogonal polynomials of one discrete variable

These polynomials satisfy a difference equation of hypergeometric type such that the difference derivatives of the some polynomial satisfy a similar equation. The discrete variable can be consider of two types.

a) *On homogeneous lattice*: $x = 0, 1, 2, \cdots$
The corresponding polynomials satisfy a difference equation

$$\sigma(x)\Delta\nabla p_n(x) + \tau(x)\Delta p_n(x) + \lambda_n p_n(x) = 0$$

with $\Delta f(x) = f(x+1) - f(x), \quad \nabla f(x) = f(x) - f(x-1)$, $\sigma(x)$ and $\tau(x)$ are functions of second and first order respectively;
an orthogonality relation

$$\sum_{x=a}^{b-1} p_n(x)\, p_m(x)\, \rho(x) = d_n^2\, \delta_{mn}$$

with $\rho(x)$a weight function and d_n a normalization constant. To these polynomials correspond the Meixner, Kravchuk, Charlier and Hahn polynomials [1]

b) *On non homogeneous lattice*: $x = x(s), \ \ s = 0, 1, 2, \cdots$ the polynomials satisfy a difference equation

$$\sigma\left[x(s)\right] \frac{\Delta}{\Delta x\left(s-\frac{1}{2}\right)} \frac{\nabla y_n(x)}{\nabla x(s)} + \frac{1}{2}\tau\left[x(s)\right] \left\{ \frac{\Delta y_n(x)}{\Delta x(s)} + \frac{\nabla y_n(x)}{\nabla x(s)} \right\} + \lambda_n y_n(x) = 0;$$

an orthogonalit relation

$$\sum_{s=a}^{b-1} y_n(s)\, y_m(s)\, \rho_n(x)\, \Delta x\left(s - \frac{1}{2}\right) = d_n^2\, \delta_{nm}.$$

The corresponding polynomials are classified according to the latice function:
For $x(s) = s(s+1)$, we have the Racach and dual Hahn polynomials
For $x(s) = q^s$ or $\frac{q^s - q^{-s}}{2}$, we have the q-Kravchuck, q-Meixner, q-Charlier and q-Hahn polynomials.
For $x(s) = \frac{q^s + q^{-s}}{2}$ or $\frac{q^{is} + q^{-is}}{2}$, we have the q-Racach and q-dual Hahn polynomials [2].

2. Generalized Clebsch-Gordon coeficients and generalized $3nj$-Wigner symbols

If two angular momentum operators are coupled to give a total angular momentum $J = J_1 + J_2$ the new basis can be expressed in terms of the old ones

$$|j_1 j_2 jm\rangle = \sum_{m_1+m_2=m} \langle j_1 j_2 m_1 m_2 \mid j_1 j_2 jm\rangle \, |j_1 j_2 m_1 m_2\rangle \, .$$

The symmetry properties of the Clebsch-Gordon coefficients in this expansion are more patent if one sbstitutes them by the Wigner symbols

$$\langle j_1 j_2 jm \mid j_1 j_2 m_1 m_2\rangle = (-1)^{j_1-j_2+j-m}\sqrt{2j+1}\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$

Similarly if we couple three angular momentum operator we obtain a new basis in terms of the old ones:

$$|j_1 j_2 j_3 j_{12} jm\rangle = \sum \langle j_1 j_2 j_3 m_1 m_2 m_3 \mid j_1 j_2 j_3 j_{12} jm\rangle \, |j_1 j_2 j_3 m_1 m_2 m_3\rangle$$

for the coupling $(J_1 + J_2) + J_3 = J$,

$$|j_1 j_2 j_3 j_{23} jm\rangle = \sum \langle j_1 j_2 j_3 m_1 m_2 m_3 \mid j_1 j_2 j_3 j_{23} jm\rangle \, |j_1 j_2 j_3 m_1 m_2 m_3\rangle$$

for the coupling $J_1 + (J_2 + J_3) = J$.

Both bases are related by some matrix $U(j_{12}, j_{23})$ that can be written in terms of generalized $6j$-Wigner symbol

$$U(j_{12}, j_{23}) = (-1)^{j_1+j_2+j_3+j}\sqrt{(2j_{12}+1)(2j_{23}+1)}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}$$

In similar fashion can be written the generalized Clebsch-Gordon coefficients and generalized $3nj$-Wigner symbols [3]. The algebraic properties of these symbols can be represented by geometrical graphs [3].

3. $3nj$-symbols as orthogonal polynomials of several discrete variable

The 6j-symbols are proportional to the Racah polynomials through the following relation [2]

$$(-1)^{j_1+j_2+j_{23}}\sqrt{(2j_{12}+1)(2j_{23}+1)}\left\{\begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{pmatrix}\right\} = \frac{\sqrt{\rho(x)}}{d_n} u_n^{(\alpha,\beta)}(x, a, b)$$

with

$x(s) = s(s+1)\ ,\ s = j_{23}$
$a = j_3 - j_2 \quad , \quad b = j + j_3 + 1 \quad , \quad n = j_{12} - j_1 + j_2 - j$
$\alpha = j_1 - j_2 - j_3 + j \quad , \quad \beta = j_1 - j_2 + j_3 - j$

Using the assymptotic limit of the Racah polynomials and the connections between the Jacobi polynomials and the Wigner little functions one can prove the following approximation of the $6j$-sumbols when $j_1 \sim j_2 \sim j_3 \sim j \gg j_{12}$

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \simeq \frac{(-1)^{j_2+j_3+j_{23}}}{\sqrt{j_1+j_2+1}\sqrt{j_3+j+1}} d^{j_{12}}_{j_1-j_2, j_3-j}(\vartheta) \tag{1}$$

with

$$\cos\theta = \frac{(2j_{23}+1)^2 - (j_1+j_2+1)^2 - (j_3+j+1)^2}{2\,(j_1+j_2+1)\,(j_3+j+1)}$$

The $3nj$-symbols of the first and second kind can be written in terms of $6j$-symbols, and therefore in terms of product of Racah polynomials, giving rise to orthogonal polynomials of several discrete variables. To illustrate this take, f.i., the $12j$-symbol of the second kind as a combination of $6j$-symbols.

$$\left\{\begin{matrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{matrix}\right\} = \sum_x (2x+1)\,(-1)^{R_n+4x} \left\{\begin{matrix} j_1 & k_1 & x \\ k_2 & j_2 & l_1 \end{matrix}\right\}$$
$$\left\{\begin{matrix} j_2 & k_2 & x \\ k_3 & j_3 & l_2 \end{matrix}\right\}\left\{\begin{matrix} j_3 & k_3 & x \\ k_4 & j_4 & l_3 \end{matrix}\right\}\left\{\begin{matrix} j_4 & k_4 & x \\ k_1 & j_1 & l_4 \end{matrix}\right\}$$

Here $R_n = \sum\limits_{i=1}^{4}(j_i + l_i + k_i)$. Substituting each 6j-symbol for the corresponding Racah polynomial we obtain:

$$\left\{\begin{matrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{matrix}\right\} = \sum_x \frac{1}{2x+1} \prod_{i=1}^{4} \frac{\sqrt{\rho\,(l_i)}}{d_{n_i}}\, u_{n_i}^{(\alpha_i,\beta_i)}\,(l_i) \equiv p_n\,(l_1\; l_2\; l_3\; l_4)$$

which is a polynomial of four discrete variables.

For the assymptotic limit we find

$$\left\{\begin{matrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{matrix}\right\} \approx \sum_x (2x+1) \prod_{i=1}^{4} \frac{1}{j_i + k_i + 1}\; d^{x}_{j_i - k_i, j_{i+1} - k_{i+1(\bmod 4)}}\,(\vartheta_i)$$

These formulas can be easily generalized to any $3nj$-symbols of first and second kind.

4. Application to spin networks and to Ponzano-Regge integral action

Penrose has proposed a model for the space and time in which the underlying structure is given by a set of interactions between elementary units that satisfy the coupling of angular momentum operators, called spin networks [4]. One particular case of these networks can be described by the graphs of $3nj$-symbols. From different point of view Regge has proposed a method to calculate Einstein action by the approximation of curved riemannian manifold by a polyedron built up of triangles. Later Ponzano and Regge applied the properties of $6j$-symbols to calculate the sum action over this triangulation [5]

Let M be a riemannian manifold that is approximated by a polyedron with boundury D and it is decomposed into p tetrahedra T_k represented by $6j$-symbols.

The polyedron give rise to triangular faces f, represented by $3j$-symbols, and to q internal edges x_i, as well as to external ones l_i with respect to the boundary D.

Ponzano and Regge define the sum

$$S = \sum_{x_i} \prod_{k=1}^{p} T_k\,(-1)^{\varphi} \prod_{i=1}^{q} (2x_i+1) \tag{2}$$

When $l_i \to \infty, \quad \hbar \to 0, \quad \hbar l_i \to \mathrm{finite}$ we recovered the continuous manifold. In order to compute the $6j$-symbols in te classical limit, we uses the assymptotic formula [2]

$$d^{j}_{mm'}(\theta) \approx (-1)^{m-m'}\sqrt{\frac{2}{\pi(j-m)}}\left(\frac{2j+m-m'+1}{2j-m+m'+1}\right)^{\frac{m+m'}{2}}$$
$$\frac{\cos\left[\left(j+\frac{1}{2}\right)\theta-\left(m-m'+\frac{1}{2}\right)\frac{\pi}{2}\right]}{\sqrt{\sin\theta}}$$

at $m \sim m' \sim 1$, $j >> 1$. Substituting this expression in (1) with $m = j_1 - j_2$, $m' = j_4 - j_5$, $j = j_6$ and taking the edges of the tetrahedra $j_1 + \frac{1}{2}, \ldots, j_6 + \frac{1}{2}$, very large except j_6, we have

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \approx \frac{1}{\sqrt{12\pi V}}\cos\left\{\left(j_6+\frac{1}{2}\right)\theta-\left(j_1+\frac{1}{2}\right)\frac{\pi}{2}+\right.$$
$$\left.+\left(j_2+\frac{1}{2}\right)\frac{\pi}{2}-\left(j_4+\frac{1}{2}\right)\frac{\pi}{2}+\left(j_5+\frac{1}{2}\right)\frac{\pi}{2}+\frac{\pi}{4}\right\}=$$
$$=\frac{1}{\sqrt{12\pi V}}\cos\left\{\sum_{i=1}^{6}\left(j_i+\frac{1}{2}\right)\theta_i+\frac{\pi}{4}\right\} \tag{3}$$

where θ_i is the dihedral angle for the edge j_i and $V = \frac{1}{6}\left(j_1+\frac{1}{2}\right)\left(j_4+\frac{1}{2}\right)\left(j_6+\frac{1}{2}\right)\operatorname{sen}\,\theta$. Note the formula (3) has been proved rigurously by Roberts [5]. Introducing formula (3) in formula (2), Ponzano and Regge proved that it leads in the continuous limit to the integral action of the general relativity.

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